International Journal of Theoretical Physics, Vol. 41, No. 4, April 2002 (© 2002)

Properties of the q-Analogue of a Squeezed Vacuum State

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Received July 10, 2001

We constructed a normalizable q-analogue of squeezed vacuum state using the technique of integration within an ordered product (IWOP) of operators and the properties of the inverses of q-deformed creation and annihilation operatots. We also study its nonclassical properties and phase probability distribution.

KEY WORDS: q-analogue; squeezed vacuum state.

1. INTRODUCTION

In the past few years, considerable attention has been focused on the deformation of the harmonic oscillator algebra of creation and annihilation operators, called the q-oscillator algebra (Arik and Coon, 1976; Chiu *et al.*, 1992; Fan and Jing, 1994, 1995; Jing and Fan, 1994; Katriel and Solomon, 1991; Kuang and Wang, 1993; Wei, 1993). From a mathematical point of view, q-oscillators are associated to the simplest nontrivial example of Hopf algebra. It should be pointed out that there are, at least, two properties which make q-oscillators interesting objects for physics. The first is the fact that they naturally appear as the basic building blocks of completely integrable theories. Therefore, in so far as complete integrability is important for physics, q-oscillators are a relevant physical tool. The second concerns the recently discovered connection between q-deformation and nonlinearity (Man'ko *et al.*, 1993a,b). Therefore, The exhaustive studies of the q-oscillator algebra and the q-deformed quantum states associated with it represent more than academic interest. Up to now, some concepts of pure states, such as the

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number state (Arik and Coon, 1976), coherent state (Chiu *et al.*, 1992), squeezed state (Katriel and Solomon, 1991), SU (1, 1) coherent state (Fan and Jing, 1995), binomial state (Jing and Fan, 1994), and even and odd coherent state (Kuang and Wang, 1993; Wei, 1993) have been extended to the q-deformed case.

In this paper, we constructed a normalizable q-analogue of squeezed vacuum state using the technique of integration within an ordered product (IWOP) of operators and the properties of the inverses of q-deformed creation and annihilation operatots. This state is different from one introduced in Fan (1994) in that it is normalizable. Finally we study its nonclassical properties and phase probability distribution.

2. THE q-ANALOGUES OF SQUEEZED VACUUM STATES

The annihilation operator a_q and the creation operator a_q^+ of q-oscillators are distortions of the annihilation and creation operators a and a^+ of the usual harmonic oscillator and are given by Arik and Coon (1976) and Chiu *et al.* (1992)

$$a_f = af(N), \qquad a_f^+ = f(N)a^+, \tag{1}$$

where

$$N = a^+a, \qquad f(N) = \sqrt{\frac{[N]}{N}}, \qquad [x] = \frac{1-q^x}{1-q} \tag{2}$$

The commutator between a_q and a_q^+ is given by

$$a_q a_q^+ - q a_q^+ a_q = 1 \tag{3}$$

We now introduce the inverse of the operators a_q and a_q^+ as follows:

$$a_q^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n+1]}} |n+1\rangle\langle n|, \tag{4}$$

$$(a_q^+)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{[n+1]}} |n\rangle\langle n+1| = \left(a_q^{-1}\right)^+ \tag{5}$$

A noncommutative relation between the inverses of the operators a_q and a_q^+ follows

$$a_q a_q^{-1} = (a_q^+)^{-1} a_q^+ = 1, \tag{6}$$

$$a_q^{-1} a_q = a_q^+ (a_q^+)^{-1} = 1 - |0\rangle\langle 0|, \tag{7}$$

which means that a_q^{-1} is the right inverse of a_q and $(a_q^+)^{-1}$ is the left inverse of a_q^+. This conclusion is analogous with the case of the inverse of ordinary boson operators. Using the inverse operators a_q^{-1}, $(a_q^+)^{-1}$, and the number operator N we define two new operators

$$A_q^+ = N a_q^{-1}, \qquad A_q = (a_q^+)^{-1} N \tag{8}$$

With the help of Eqs. (7) and (8), we have

$$\begin{aligned}[A_q, a_q^+] &= (a_q^+)^{-1} N a_q^+ - a_q^+ (a_q^+)^{-1} N \\ &= (a_q^+)^{-1} a_q^+ (N+1) - (1 - |0\rangle\langle 0|) N = 1\end{aligned} \tag{9}$$

A_q is thus a canonically conjugate to a_q^+. Following essentially the same analysis as in Fan (1994), we can prove that the normal product form of the vacuum projector is

$$|0\rangle\langle 0| =: \exp(-a_q^+ A_q) :, \tag{10}$$

here the normal ordering : : is for (a_q^+, A_q), which is absolutely different from the normal product form of the vacuum projector $|0\rangle\langle 0|$ in Fan *et al.* (1987) where the normal ordering is for (a^+, a). With the help of Eq. (10), we can prove the following overcompleteness relation composed by $\|z\rangle = \exp(z a_q^+)|0\rangle$ and $\langle z| = \langle 0| \exp(z^* A_q)$

$$\int \frac{d^2 z}{\pi} \exp(-|z|^2) \|z\rangle\langle z| = \int \frac{d^2 z}{\pi} : \exp(-|z|^2 + z a_q^+ + z^* A_q - a_q^+ A_q) := 1 \tag{11}$$

Letting $\hat{x}' = (A_f + a_f^+)/\sqrt{2}$ we have the q-analogue of a coordinate state

$$\|x'\rangle = \pi^{-1/4} \exp\left[-\frac{1}{2} x'^2 + \sqrt{2} x' a_f^+ - \frac{1}{2} a_f^{+^2}\right] |0\rangle, \tag{12}$$

$$\langle x'| = \langle 0| \exp\left(-\frac{1}{2} x'^2 + \sqrt{2} x' A_f - \frac{1}{2} A_f^2\right) \pi^{-1/4}, \tag{13}$$

$$\hat{x}' \|x'\rangle = x' \|x'\rangle, \qquad \langle x'| \hat{x}' = \langle x'| x' \tag{14}$$

Performing the following integration by virtue of the IWOP technique we obtain the q-analogue of a squeeze operator

$$\begin{aligned}S' &= \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{\mu}} \left\| \frac{x'}{\mu} \right\rangle \langle x'| \\ &= \exp\left(-\frac{1}{2} a_q^{+^2} \tanh\lambda\right) \exp\left[\left(N + \frac{1}{2}\right) \ln \sec h\lambda\right] \exp\left(\frac{1}{2} A_q^2 \tanh\lambda\right),\end{aligned} \tag{15}$$

where $\mu = \exp\lambda$, $\tanh\lambda = (\mu^2 - 1)/(\mu^2 + 1)$. From Eqs. (15) we can prove that S' engenders the following squeeze transformations:

$$S' A_q S'^{-1} = A_q \cosh\lambda + a_q^+ \sinh\lambda, \tag{16}$$

$$S' a_q^+ S'^{-1} = a_q^+ \cosh\lambda + A_q \sinh\lambda \tag{17}$$

Operating S' on the vacuum state $|0\rangle$ we obtain the q-analogue of a squeezed vacuum state

$$\begin{aligned} |\lambda, q\rangle &= C_0 \exp\left(-\frac{1}{2} a_f^{+^2} \tanh\lambda\right) |0\rangle \\ &= C_0 \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}\tanh\lambda\right)^n a_q^{+^{2n}} |0\rangle \\ &= C_0 \sum_{n=0}^{\infty} \left(-\frac{1}{2}\tanh\lambda\right)^n \frac{\sqrt{[2n]!}}{n!} |2n\rangle, \end{aligned} \tag{18}$$

where

$$[k]! = [1][2]\cdots[k], \qquad [0]! = 1, \tag{19}$$

and the normalization constant C_0 is given by

$$C_0 = \left[\sum_{n=0}^{\infty} \frac{\tanh^{2n}\lambda}{2^{2n}} \frac{[2n]!}{(n!)^2}\right]^{-1/2} \tag{20}$$

From Eqs. (2) and (19), we can see that $[2n]! \leq (2n)!$. Since the series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\tanh\lambda\right)^{2n} \frac{(2n)!}{(n!)^2}$$

is convergent, the series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\tanh\lambda\right)^{2n} \frac{[2n]!}{(n!)^2}$$

is also convergent. Therefore, the state $|\lambda, q\rangle$ given by Eq. (19) is normalizable, which is quite different from one introduced in Fan (1994).

3. THE NONCLASSICAL PROPERTIES OF THE STATE $|\lambda, q\rangle$

We now calculate the dispersions of the electromagnetic field, assumed to be expressed in terms of the conventional operators a, a^+ in the standard way

$$X_1 = \frac{1}{\sqrt{2}}(a + a^+), \qquad X_2 = \frac{1}{i\sqrt{2}}(a - a^+) \tag{21}$$

It then follows

$$\begin{aligned} (\Delta X_1)^2 &= \left\langle X_1^2 \right\rangle - \langle X_1 \rangle^2 \\ &= \frac{1}{2} + \langle a^+ a\rangle + \mathrm{Re}\left\langle a^{+^2}\right\rangle - 2(\mathrm{Re}\,\langle a^+\rangle)^2, \end{aligned} \tag{22}$$

$$\begin{aligned}(\Delta X_2)^2 &= \left\langle X_2^2 \right\rangle - \langle X_2 \rangle^2 \\ &= \frac{1}{2} + \langle a^+ a \rangle - \mathrm{Re}\left\langle a^{+^2} \right\rangle - 2(\mathrm{Im}\,\langle a^+ \rangle)^2 \end{aligned} \tag{23}$$

If

$$G_i = 4\langle(\Delta X_i)^2\rangle - 1 < 0, \quad (i = 1, 2), \tag{24}$$

then the X_i component of the field is squeezed. Maximum (100%) squeezing is obtained for $G_i = -1$.

From Eq. (18) we obtain the expectation values of the corresponding quantities in the state $|\lambda, q\rangle$ as follows:

$$\langle a^+ \rangle = 0, \tag{25}$$

$$\langle a^+ a \rangle = \sum_{n=0}^{\infty} 2n F_n^2, \tag{26}$$

$$\left\langle a^{+^2} \right\rangle = \sum_{n=0}^{\infty} \sqrt{(2n+1)(2n+2)} F_n F_{n+1}, \tag{27}$$

where

$$F_n = C_0 \left(-\frac{1}{2} \tanh \lambda \right)^n \frac{\sqrt{[2n]!}}{n!} \tag{28}$$

The numerical calculation results of G_1 for various values of the parameter q are presented in Fig. 1. From these figures we can see that the degree of squeezing increases with increasing q, which is similar to the squeezing property of the q-squeezed state corresponding to $\mathrm{SU}_q\,(1, 1)$ (Solomon and Katriel, 1990). Moreover, the numerical calculation results of the second-order correlation function $g^{(2)}(0) = \langle a^{+^2} a^2 \rangle / \langle a^+ a \rangle^2$ show that like the usual squeezed vacuum state, the photon in the state $|\lambda, q\rangle$ is bunching.

4. PHASE PROBABILITY DISTRIBUTION OF THE STATE $|\lambda, q\rangle$

We now turn to the phase probability distributions for the state $|\lambda, q\rangle$. According to the Pegg–Barnett formalism (Barnett and Pegg, 1989; Pegg and Barnett, 1988) we start with a finite dimensional $(s + 1)$ Hilbert space spanned by the number states $|0\rangle, |1\rangle, \ldots, |s\rangle$. In this space a complete orthonormal set of phase states $|\theta_m\rangle$, $m = 0, 1, \ldots, s$, is defined by

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(in\,\theta_m)|n\rangle, \tag{29}$$

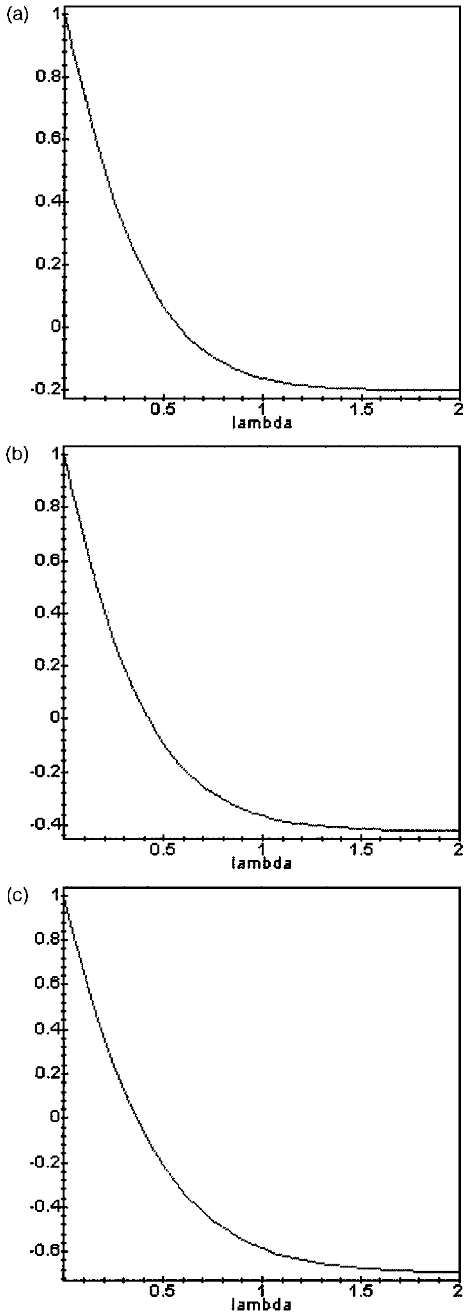

Fig. 1. (a) G_1 versus λ for $q = 0.1$; (b) G_1 versus λ for $q = 0.6$; (c) G_1 versus λ for $q = 0.9$.

where

$$\theta_m = \theta_0 + \frac{2\pi m}{s+1}, \quad m = 0, 1, \ldots, s \tag{30}$$

The value of θ_0 is arbitrary and defines a particular basis in the phase space. A hermitian phase operator Φ_θ is defined as

$$\Phi_\theta = \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m| \tag{31}$$

For the state $|\lambda, q\rangle$, the phase probability distribution is given by

$$|\langle\theta_m \mid \lambda,\ f\rangle|^2 = \frac{1}{s+1} + \frac{2}{s+1} \sum_{n>k} F_n F_k \cos[2(n-k)\theta_m] \tag{32}$$

Choosing θ_0 as

$$\theta_0 = -\frac{\pi s}{s+1} \tag{33}$$

we have from Eq. (31)

$$|\langle\theta_m \mid \lambda,\ f\rangle|^2 = \frac{1}{s+1} + \frac{2}{s+1} \sum_{n>k} F_n F_k \cos 2(n-k) \frac{2\pi\mu}{s+1} \tag{34}$$

where $\mu = m - s/2$. The continuous phase probability distribution can be obtained as

$$\begin{aligned} P(\theta) &= \lim_{s\to\infty} \frac{s+1}{2\pi} |\langle\theta_m \mid \lambda,\ f\rangle|^2 \\ &= \frac{1}{2\pi} \left(1 + 2 \sum_{n>k} F_n F_k \cos[2(n-k)\theta] \right) \end{aligned} \tag{35}$$

The results of numerical computations of the continuous phase probability distribution for the state $|\lambda,\ f\rangle_2$ are presented in Fig. 2. It can be easily seen that the height of two peaks increases with increasing q.

5. SUMMARY

In this paper, we constructed a normalizable q-analogue of squeezed vacuum state using the technique of integration within an ordered product (IWOP) of operators and the properties of the inverses of q-deformed creation and annihilation operators. This state can exhibit squeezing effects and the degree of squeezing increases with increasing q. No antibunching effect is found. We also study its phase probability distribution using the Pegg–Barnett formalism. The results show that the parameter q only affects the height of the peaks.

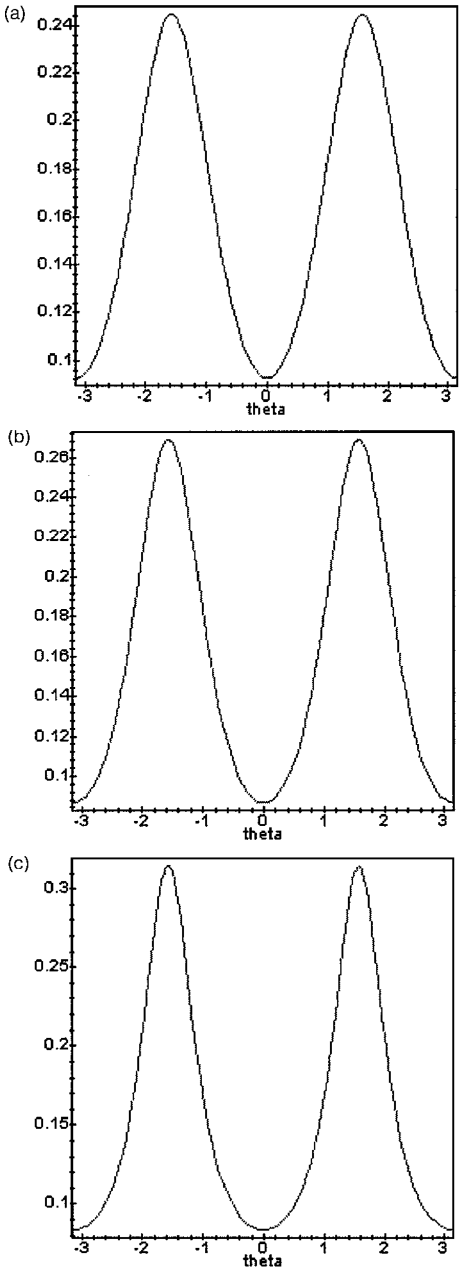

Fig. 2. (a) Phase probability distribution of the state $|\lambda, q\rangle$; (b) Phase probability distribution of the state $|\lambda, q\rangle$ for $\lambda = 0.5$, $q = 0.5$; (c) Phase probability distribution of the state $|\lambda, q\rangle$ for $\lambda = 0.5$, $q = 0.9$.

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